JORDAN GROUPS AND ALGEBRAIC SURFACES

TATIANA BANDMAN AND YURI G. ZARHIN

Abstract. We prove that an analogue of Jordan's theorem on finite subgroups of general linear groups holds for the groups of biregular automorphisms of algebraic surfaces. This gives a positive answer to a question of Vladimir L. Popov.

1. Introduction

Throughout this paper, k is an algebraically closed field of characteristic zero and $\mathbb{P}^1$ is the projective line over k. Let U be an *algebraic variety* over k [14, Vol. 2, Ch. VI, Sect. 1]. Then $U(k)$ and $\mathrm{Aut}(U)$ stand for its set of k-points and the group of biregular k-automorphisms respectively. Unless otherwise stated, by a point of U we mean a k-point. If U is *irreducible* then we write $k(U)$ and $\mathrm{Bir}(U)$ for its field of rational functions and the group of birational k-automorphisms respectively; $\mathrm{Aut}(U)$ is a subgroup of $\mathrm{Bir}(U)$. By an elliptic curve we mean an irreducible smooth projective curve of genus 1 over k. If X is an elliptic curve and $\mathcal{T} \subset X(k)$ is a *nonempty* finite set of points on X then the (sub)group

$$\mathrm{Aut}(X,\mathcal{T}) = \{u \in \mathrm{Aut}(X) \mid u(\mathcal{T}) = \mathcal{T}\} \subset \mathrm{Aut}(X)$$

is *finite*, since $X \setminus \mathcal{T}$ is a *hyperbolic* curve. If $\mathcal{S}$ is a smooth irreducible projective surface over k then an irreducible closed curve C in $\mathcal{S}$ is called a (-1)-curve if it is smooth rational and its self-intersection index is -1.

The following definition was inspired by the classical theorem of Jordan [2, Sect. 36] about finite subgroups of general linear groups over fields of characteristic zero.

Definition 1.1 (Definition 2.1 of [9])**.** A group B is called a *Jordan group* if there exists a positive integer J_B such that every finite subgroup B_1 of B contains a normal commutative subgroup, whose index in B_1 is at most J_B.

Remark 1.2. Clearly, a subgroup of a Jordan group is also Jordan. If a Jordan group G_1 is a subgroup of *finite* index in a group G then G is also Jordan.

V. L. Popov ([9, Sect. 2], see also [10]) posed a question whether $\mathrm{Aut}(S)$ is a Jordan group when S is an algebraic surface over k. He obtained a positive answer to his question for almost all surfaces. (The case of rational surfaces was treated earlier by J.-P. Serre [12, Sect. 5.4]). The only remaining case is when S is birationally (but not biregularly) isomorphic to a product $X \times \mathbb{P}^1$ of an elliptic curve X and the projective line. In [16] the second named author proved that $\mathrm{Aut}(S)$ is

2010 *Mathematics Subject Classification.* 14E07, 14J26, 14J50, 14L30, 14K05.

The first named author is partially supported by the Ministry of Absorption (Israel), the Israeli Science Foundation (Israeli Academy of Sciences, Center of Excellence Program), the Minerva Foundation (Emmy Noether Research Institute of Mathematics).

The second named author is partially supported by a grant from the Simons Foundation (#246625 to Yuri Zarkhin).

a Jordan group if S is a *projective* surface. The aim of this paper is to extend this result to the case of arbitrary algebraic surfaces. Our main result is the following statement, which gives a positive answer to Popov's question.

Theorem 1.3. *If X is an elliptic curve over k and S is an irreducible normal algebraic surface that is birationally isomorphic to $X \times \mathbb{P}^1$ then $\mathrm{Aut}(S)$ is a Jordan group.*

Remark 1.4. The group $\mathrm{Bir}(X \times \mathbb{P}^1)$ is *not* Jordan [15].

Remark 1.5. Suppose that S is a *non*-smooth irreducible normal surface. Since it is normal, there are only finitely many singular points on S. Then, by [10, Sect. 2, Cor. 8], $\mathrm{Aut}(S)$ is Jordan. This implies that in the course of the proof of Theorem 1.3 we may assume that S is smooth. On the other hand, by a theorem of Zariski [17, Cor. II.2.6 on p. 53], every irreducible smooth surface is quasi-projective. This implies that in the course of the proof of Theorem 1.3 we may assume that S is *smooth quasi-projective*.

Corollary 1.6. *Suppose that V is an irreducible normal algebraic variety over k. If $\dim(V) \le 2$ then $\mathrm{Aut}(V)$ is Jordan.*

Proof of Corollary 1.6. We have $\mathrm{Aut}(V) \subset \mathrm{Bir}(V)$. If V is *not* birationally isomorphic to a product of the projective line and an elliptic curve then $\mathrm{Bir}(V)$ is Jordan ([9, Th. 2.32]) and therefore its subgroup $\mathrm{Aut}(V)$ is also Jordan. If V is birationally isomorphic to a product of the projective line and an elliptic curve then $\dim(V) = 2$ and Theorem 1.3 implies that $\mathrm{Aut}(V)$ is Jordan. □

Theorem 1.7. *Let V be an irreducible algebraic variety over k. If $\dim(V) \le 2$ then $\mathrm{Aut}(V)$ is Jordan.*

Proof of Theorem 1.7. Let $\nu : V^{\nu} \to V$ be the *normalization* of V ([8, Ch. III, Sect. 8], [4, Ch. 2, Sect. 2.14]). Here ν is a birational (surjective) regular map that is called the normalization map for V and V^{ν} is an irreducible *normal* variety (of the same dimension as V) over k [8, Th. 4 on p. 203]. The *universality property* of the normalization map implies that every biregular automorphism of V lifts uniquely to a biregular automorphism of V^{ν} [4, Ch. 2, Sect. 2.14, Th. 2.25 on p, 141]. This give rise to the *embedding* of groups

$$\mathrm{Aut}(V) \hookrightarrow \mathrm{Aut}(V^{\nu}).$$

By Corollary 1.6, the group $\mathrm{Aut}(V^{\nu})$ is Jordan. Since $\mathrm{Aut}(V)$ is isomorphic to a subgroup of Jordan $\mathrm{Aut}(V^{\nu})$, it is also Jordan.

□

Corollary 1.8. *Let V be an algebraic variety over k. If $\dim(V) \le 2$ then $\mathrm{Aut}(V)$ is Jordan.*

Proof. Let $V_1, \dots, V_r$ be all the *irreducible* components of V. Clearly, all V_i are irreducible algebraic varieties with $\dim(V_i) \le \dim(V) \le 2$. By Theorem 1.7, all $\mathrm{Aut}(V_i)$ are Jordan. Now Lemma 1 in Section 2.2 of [10] implies that $\mathrm{Aut}(V)$ is also Jordan. □

Remark 1.9. Suppose that k is the field $\mathbb{C}$ of complex numbers and X is a smooth irreducible quasi-projective non-projective surface. Then $M = X(\mathbb{C})$ carries the natural structure of a connected oriented smooth *real noncompact* fourfold and the group $\mathrm{Aut}(X)$ embeds naturally in the group $\mathrm{Diff}(M)$ of the (real) diffeomorphisms of the fourfold M. While $\mathrm{Aut}(X)$ is always Jordan, there are examples of connected oriented smooth *noncompact* real fourfolds, whose group of diffeomorphisms is *not* Jordan [11].

The paper is organized as follows. In Section 2 we discuss *minimal closures* of surfaces. In Section 3 we prove Theorem 1.3.

Acknowledgements. We are deeply grateful to Vladimir Popov for a stimulating question and useful discussions. This work was started in September 2013 when both authors were visitors at the Max-Planck-Institut für Mathematik (Bonn), whose hospitality and support are gratefully acknowledged. Most of this work was done during the academic year 2013/2014 when the second named author (Y.Z.) was Erna and Jakob Michael Visiting Professor in the Department of Mathematics at the Weizmann Institute of Science, whose hospitality and support are gratefully acknowledged.

2. Minimal closures

2.1. Let X be an elliptic curve over k and S be a *smooth* irreducible quasi-projective surface over k that is birationally isomorphic to $X \times \mathbb{P}^1$. There exists an irreducible smooth projective surface $\bar{S}$ such that its certain Zariski-open subset is biregularly isomorphic to S (further we identify S with this open subset). Clearly, the inclusion map $S \subset \bar{S}$ is a birational morphism. This implies that

$$\mathrm{Aut}(S) \subset \mathrm{Bir}(S) = \mathrm{Bir}(\bar{S})$$

and therefore one may view $\mathrm{Aut}(S)$ as a subgroup of $\mathrm{Bir}(\bar{S})$. Since $\bar{S}$ is birationally isomorphic to S, it also birationally isomorphic to $X \times \mathbb{P}^1$.

Let us fix a birational isomorphism between $\bar{S}$ and $X \times \mathbb{P}^1$. The projection map $X \times \mathbb{P}^1 \to X$ gives rise to a rational map $\bar{\pi} : \bar{S} \to X$ with dense image. Since $\bar{S}$ is smooth and X becomes abelian variety (after a choice of a base point), it follows from a theorem of Weil [1, Sect. 4.4] that $\bar{\pi}$ is regular. Since $\bar{S}$ is projective, $\bar{\pi} : \bar{S} \to X$ is surjective, because its image is closed.

For each $x \in X(k)$ we write $\bar{F}_x$ for the effective divisor $\bar{\pi}^*(x)$ on $\bar{S}$ that is the pullback (under $\bar{\pi}$) of the divisor (x) on $\bar{S}$. Clearly, the support of $\bar{F}_x$ coincides with the curve $\bar{\pi}^{-1}(x)$ on $\bar{S}$. One say that the fiber of $\bar{\pi}$ over x is *reduced* if all irreducible components of the divisor $\bar{F}_x$ have multiplicity 1. We say that the fiber of $\bar{\pi}$ over x is *irreducible* if the curve $\bar{\pi}^{-1}(x)$ is irreducible; if this is the case then its multiplicity in $\bar{F}_x$ is 1 [6, Ch. 3, Sect. 1.4, Lemma 1.4.1(1) on p. 195].

It is known [13, Ch. IV] that for all but finitely many $x \in X(k)$ the fiber of $\bar{\pi}$ over x is irreducible and reduced, and the curve $\bar{\pi}^{-1}(x)$ is smooth (and irreducible). We call such fibers nonsingular and other fibers *singular*.

If C is a rational curve on $\bar{S}$ then the restriction of $\bar{\pi}$ to C must be a constant map, because every map from a rational curve to an elliptic curve is constant. This implies that C lies in a fiber of $\bar{\pi}$. (In particular, every (-1)-curve on $\bar{S}$ lies in a fiber of $\bar{\pi}$.) This implies that every birational automorphism of $\bar{S}$ is fiberwise [5, Sect. 13, Th. 2]; see Sect. 2.2 below.

However, if $x \in X(k)$ and the fiber $\bar{\pi}^{-1}(x)$ is singular then the corresponding divisor $\bar{F}_x$ enjoys the following properties [6, Ch. I, Sect. 2.12; Ch. 3, Sect. 1.4, Lemma 1.4.1 on p. 195]] (see also [3]).

(i) Each irreducible component of $\bar{F}_x$ is a smooth rational curve (and the corresponding graph is a tree) [3, Sect. 3].
(ii) At least, one of the irreducible components of $\bar{F}_x$ is a (-1)-curve [3, Sect. 4.2].
(iii) If one of the irreducible components of $\bar{F}_x$ is a (-1)-curve of multiplicity 1 then there is another irreducible (-1)-component of $\bar{F}_x$ ([3, Sect. 4.2].

2.2. If $\sigma \in \mathrm{Bir}(\bar{S})$ then there is a unique *biregular* automorphism $\mathfrak{f}(\sigma) : X \to X$ such that the composition $\bar{\pi}\sigma$ is a *regular* map that coincides with the composition

$$\mathfrak{f}(\sigma)\bar{\pi} : \bar{S} \stackrel{\bar{\pi}}{\to} X \stackrel{\mathfrak{f}(\sigma)}{\to} X$$

(see, e.g., [7, Lecture V, Sect. 1.4, p. 99]). Clearly, σ sends the fiber $\bar{\pi}^{-1}(x)$ to the fiber $\bar{\pi}^{-1}(\mathfrak{f}(\sigma)(x))$ for all $x \in X(k)$. We get a surjective group homomorphism

$$\mathfrak{f} : \mathrm{Bir}(\bar{S}) \to \mathrm{Aut}(X),\ \sigma \mapsto \mathfrak{f}(\sigma)$$

that fits into a short exact sequence

$$\{1\} \to \mathrm{Bir}_X(\bar{S}) \subset \mathrm{Bir}(\bar{S}) \stackrel{\mathfrak{f}}{\to} \mathrm{Aut}(X) \to \{1\}$$

where the subgroup $\mathrm{Bir}_X(\bar{S})$ consists of all birational automorphisms $\sigma \in \mathrm{Bir}(\bar{S})$ such that $\bar{\pi}\sigma = \bar{\pi}$ (i.e. σ leaves invariant every fiber of $\bar{\pi}$). In addition, $\mathrm{Bir}_X(\bar{S})$ is isomorphic to the projective linear group $\mathrm{PGL}(2, k(X))$ over the field $k(X)$ of rational functions on X [7, Lecture V, Sect. 1.4, p. 99].

2.3. We write π for the composition

$$S \subset \bar{S} \stackrel{\bar{\pi}}{\to} X,$$

i.e., for the restriction of π to S. Recall that $\mathrm{Aut}(S) \subset \mathrm{Bir}(\bar{S})$. Since S is a surface, it is not contained in a union of finitely many fibers of π in $\bar{S}$. This implies that $\pi(S)$ is infinite and therefore is everywhere dense in X. It follows from [14, vol. 1, Ch. 1, Sect. 5, Th. 6] that either $\pi(S) = X$ or the complement $T_0 := X(k) \setminus \pi(S(k))$ is a finite set and

$$S \subset \pi^{-1}(X \setminus T_0) \subset \bar{S}.$$

If we write $\mathrm{Aut}_X(S)$ for the intersection (in $\mathrm{Bir}(\bar{S})$) of $\mathrm{Aut}(S)$ and $\mathrm{Bir}_X(\bar{S})$ then we get a short exact sequence

$$\{1\} \to \mathrm{Aut}_X(S) \subset \mathrm{Aut}(S) \stackrel{\mathfrak{f}}{\to} f(\mathrm{Aut}(S)) \to \{1\}$$

where

$$\mathrm{Aut}_X(S) \subset \mathrm{Bir}_X(\bar{S}),\ \mathfrak{f}(\mathrm{Aut}(S)) \subset \mathrm{Aut}(X).$$

Similarly to the case of projective surfaces, if $x \in X(k)$ then we write F_x for the effective divisor $\pi^*(x)$ on S that is the pullback (under π) of the divisor (x) on S. Clearly, the support of F_x coincides with the curve $\pi^{-1}(x)$ on S. It is also clear that the divisor F_x on S is the pullback of the divisor $\bar{F}_x$ on $\bar{S}$ under the (open) inclusion map $S \subset \bar{S}$. One says that the fiber of π over x is *reduced* if all irreducible components of the divisor F_x have multiplicity 1. We say that the fiber of π over x is irreducible if it is a multiple of a *simple* divisor, i.e., the curve $\bar{\pi}^{-1}(x)$ is irreducible. Clearly, if the fiber of $\bar{\pi}$ over x is irreducible (resp. reduced, resp. smooth) then the fiber of π over x is irreducible (resp. reduced, resp. smooth). On the other hand, if $\bar{F}_x$ has an irreducible component, say, $\bar{C}$ that appears in $\bar{F}_x$ with multiplicity $m > 1$ and, in addition, $\bar{C}$ meets S then $C := \bar{C} \bigcap S$ is an irreducible curve in S that is a component of F_x and that appears in F_x with the same multiplicity

m; in particular, the fiber of π over x is *not* reduced. Notice also that if $\bar{C}_1$ and $\bar{C}_2$ are distinct irreducible components of $\bar{F}_x$ that meet F_x then $C_1 := \bar{C}_1 \bigcap S$ and $C_2 := \bar{C}_2 \bigcap S$ are *distinct* irreducible components of F_x; in particular, the fiber of π over x is *not* irreducible.

It follows from the results about the fibers of $\bar{\pi}$ mentioned in Sect. 2.1 (see also theorems of Bertini [14, vol. 1, Ch. 2, Sect. 6.1 and 6.2] that either all the fibers of π are smooth irreducible reduced or the set T_1 of points $x \in \pi(S(k)) \subset X(k)$ such that, at least, one of these properties does not hold, is finite. Clearly,

$$\mathfrak{f}(\mathrm{Aut}(S)) \subset \mathrm{Aut}(X, T_0), \ \mathfrak{f}(\mathrm{Aut}(S)) \subset \mathrm{Aut}(X, T_1).$$

This implies that if either T_0 or T_1 is *non*-empty then $\mathfrak{f}(\mathrm{Aut}(S))$ is a *finite* group and $\mathrm{Aut}_X(\bar{S})$ is a subgroup of *finite index* in $\mathrm{Aut}(S)$.

2.4. It follows from the theorem of Jordan that the projective linear group $\mathrm{PGL}(2, k(X))$ is Jordan [9, 16]. Since $\mathrm{Bir}_X(\bar{S})$ is isomorphic to $\mathrm{PGL}(2, k(X))$ (see Sect. 2.2), it is also a Jordan group. This implies in turn that its subgroup $\mathrm{Aut}_X(S)$ is also Jordan. It follows that if either T_0 or T_1 is *non*-empty then $\mathrm{Aut}(S)$ contains the Jordan subgroup $\mathrm{Aut}_X(S)$ of finite index and therefore is Jordan itself, thanks to Remark 1.2.

In order to handle the case of empty T_0 and T_1, we need additional ideas.

Definition 2.5. The projective surface $\bar{S}$ is called a (relative) *minimal closure* of S if every (-1)-curve on $\bar{S}$ meets S. See [3, Sect. 4.9]. A minimal closure of S always exists [3, Prop. 4.10]. (Warning: if $\bar{S}$ is a minimal closure then the complement of S in $\bar{S}$ does *not* have to be a divisor!)

Lemma 2.6 (Lemma 4.12 of [3])**.** *Assume that $\pi(S) = X$ and all the fibers of π are smooth irreducible and reduced.*

If $\bar{S}$ is a minimal closure of S then all the fibers of $\bar{\pi} : \bar{S} \to X$ are irreducible.

Proof. Suppose that there exists $x \in X(k)$ such that the fiber of $\bar{\pi}$ over x is not irreducible and therefore is singular. Then $\bar{F}_x$ contains as an irreducible component a (-1)-curve, say $\bar{C}_1$ with multiplicity $m \geq 1$ (Sect. 2.1). The minimality of $\bar{S}$ implies that $C_1 = \bar{C}_1 \bigcap S$ is non-empty and therefore is an irreducible component of F_x with the same multiplicity m (Sect. 2.3). Since the fiber of π over x is reduced, $m = 1$. This implies that $\bar{F}_x$ contains another irreducible component $\bar{C}_2$ that is also a (-1)-curve. Again $C_2 = \bar{C}_2 \bigcap S$ is an irreducible component of F_x that does not coincide with C_1. This implies that the fiber of π over x is *not* irreducible, which is not the case. □

Theorem 2.7. *Assume that $\pi(S) = X$ and all the fibers of π are smooth irreducible and reduced. Let $\bar{S}$ be a minimal closure of S Then every biregular automorphism of S extends uniquely to a biregular automorphism of $\bar{S}$. In other words,*

$$\mathrm{Aut}(S) \subset \mathrm{Aut}(\bar{S}) \subset \mathrm{Bir}(\bar{S}).$$

Proof. By Lemma 2.6, every fiber $\bar{F}_x$ is an irreducible curve isomorphic to $\mathbb{P}^1$.

Let $g : S \to S$ be a biregular automorphism of S. Let us extend g to a birational map

$$\bar{g} : \bar{S} \to \bar{S}.$$

Assume that $\bar{g}$ is *not* a regular map. Let S' be a *resolution of the indeterminacies* of $\bar{g}$, i.e. a smooth irreducible surface included into the following commutative digram.

$$\begin{array}{cccc} S' & & & \\ u\downarrow & \searrow g' & & \\ \bar{S} & \overset{\bar{g}}{\dashrightarrow} \bar{S} & & \\ \cup & \cup & & , \\ S & \xrightarrow{g} S & & \\ \pi\downarrow & \downarrow & \pi & \\ X & \xrightarrow{h} X & & \end{array}$$

where u is a birational morphism that is a composition of finitely many blow ups and induces a biregular isomorphism between $u^{-1}(S)$ and S (such an u exists, because g is defined on S), g' and $\bar{\pi}' = \bar{\pi} \circ u$ are morphisms, and $h = \mathfrak{f}(g) \in \mathrm{Aut}(X)$ is a biregular automorphism of X. (The group homomorphism $\mathfrak{f}$ is defined in Sect. 2.2.) Let $D' \subset S'$ be the union of all exceptional curves for g' and let $D = g'(D') \subset \bar{S}$, which is a finite set.

Every point z of $\bar{S}$ that does not lie on D has only one preimage ${g'}^{-1}(z) \in S'$ ([14, Ch. 2, Sect. 4, Th. 2]*).*

Let B' be the union of exceptional curves for u. Clearly,

$$B' \subset S' \setminus u^{-1}(S).$$

This implies that

$$u(B') \bigcap S = \emptyset.$$

We want to show that $B' \subset D'$, because then one may contract all components of B' and $\bar{g}$ would appear to be a morphism.

Let C' be an irreducible component of B'. The *point* $u(C')$ lies in $u(B')$ and therefore does *not* belong to S.

Since X is an elliptic curve, and C' is rational, $\bar{\pi}(g'(C'))$ is a point $x \in X(k)$. Thus, since all the fibers of $\bar{\pi}$ are irreducible (thanks to Lemma 2.6), either

Case 1. $g'(C')$ is a point and therefore $C' \subset D'$;

or

Case 2. $g'(C') = \bar{F}_x = \bar{\pi}^{-1}(x) \subset \bar{S}$. Let us put $x_1 := h^{-1}(x) \in X(k)$. Then $x = h(x_1) \in X(k)$. Let $s \in F_x \setminus (F_x \cap D) \subset S$ be a point of the fiber F_x, which is not in the image of D'. Therefore it has only one *preimage* $s_1 := {g'}^{-1}(s)$. Moreover, $s_1 \in u^{-1}(S)$, because $s \in S$. On the other hand, since $g'(C') = \bar{F}_x$, there is a point $c \in C' \subset S' \setminus u^{-1}(S)$ such that $g'(c) = s$. Clearly, $c \neq s_1$ and we get a contradiction that shows that the Case 2 does not occur.

This proves that every $g \in \mathrm{Aut}(S)$ extends to a regular birational map $\bar{g} : \bar{S} \to \bar{S}$. Since the same is true for $g^{-1} \in \mathrm{Aut}(S)$, the map $\bar{g}$ is a biregular automorphism of $\bar{S}$.

□

3. Proof of Theorem 1.3

Remark 1.5 tells us that we may assume that S is a smooth quasi-projective surface. In light of results of Section 2.4, we may also assume that every fiber of π is smooth irreducible and reduced, and $\pi(S) = X$. Let $\bar{S}$ be a minimal closure of S.

By Theorem 2.7, $\mathrm{Aut}(S)$ is a subgroup of $\mathrm{Aut}(\bar{S})$. Since $\bar{S}$ is projective, the results of [16] imply that the group $\mathrm{Aut}(\bar{S})$ is Jordan and therefore its every subgroup is Jordan. It follows that $\mathrm{Aut}(S)$ is Jordan.

References

[1] S. Bosch, W. Lütkebohmert, M. Raynaud, Néron Models. Springer-Verlag, Berlin Heidelberg New York 1990.

[2] C.W. Curtis, I.Reiner, Representation Theory of Finite Groups and Associative Algebras, Wiley, New York, 1962.

[3] T. Fujita *On the topology of non-complete algebraic surfaces* . J. Fac. Sci. Univ. Tokyo, Sect. IA **29** (1982), 503-566.

[4] Sh. Iitaka, Algebraic Geometry, GTM 76. Springer-Verlag, Berlin Heidelberg New York, 1982.

[5] V. A. Iskovskikh and I. R. Shafarevich, *Algebraic Surfaces*. In: Algebraic Geometry II, pp. 127–254. Encyclopaedia of Mathematical Sciences, Vol. **35**, Springer–Verlag, 1996.

[6] M. Miyanishi, Open Algebraic Surfaces, CRM Monograph Series, **12**, American Mathematical Society, Providence, RI, 2001.

[7] J. Miyaoka, Th. Peternell, Geometry of Higher Dimensional Algebraic Varieties. DMV Seminar, Bd. **26**, Bikhäuser, Basel 1997.

[8] D. Mumford, The Red Book of Varieties and Schemes. Lecture Notes in Math. vol. **1358**, Springer, 1999.

[9] V.L. Popov, *On the Makar-Limanov, Derksen invariants, and finite automorphism groups of algebraic varieties*. In: Affine Algebraic Geometry (The Russell Festschrift), CRM Proceedings and Lecture Notes **54**, American Mathematical Society, Providence, RI, 2011, pp. 289–311.

[10] V.L. Popov, *Jordan groups and automorphism groups of algebraic varieties*. In: Groups of automorphisms in Birational and Affine Geometry, Springer Lecture Notes in Math. (2014), to appear; arXiv:1307.5522

[11] V.L. Popov, *Finite subgroups of diffeomorphism groups*. arXiv: 1310.6548 [math.GR] .

[12] J-P. Serre, *A Minkowski-style bound for the orders of the finite subgroups of the Cremona group of rank 2 over an arbitrary field*. Moscow Math. J. **9** (2009), no. 1, 183–198.

[13] I.R. Shafarevich et al., Algebraic Surfaces. Proc. Steklov Inst. Math. **75**, Moscow, 1965; American Mathematical Society, Providence, RI, 1967.

[14] I.R. Shafarevich, Basic Algebraic Geometry, second edition, Vol. 1 and 2. Springer Verlag, Berlin Heidelberg New York, 1994.

[15] Yu.G. Zarhin, *Theta groups and products of abelian and rational varieties*. Proc. Edinburgh Math. Soc. **57**, issue 1 (2014), 299–304.

[16] Yu.G. Zarhin, *Jordan groups and elliptic ruled surfaces*. Transformation Groups, to appear; arXiv:1401.7596 [math.AG] .

[17] O. Zariski, Introduction to the Problem of Minimal Models in the Theory of Algebraic Surfaces. Publ. Math. Soc. of Japan **4**, Tokyo, 1958.

Department of Mathematics, Bar-Ilan University, 5290002, Ramat Gan, ISRAEL
E-mail address: bandman@macs.biu.ac.il

Department of Mathematics, Pennsylvania State University, University Park, PA 16802, USA

Department of Mathematics, The Weizmann Institute of Science, POB 26, Rehovot 7610001, Israel
E-mail address: zarhin@math.psu.edu